

The University of British Columbia

Midterm 1 Solutions - October 2, 2014

Mathematics 105

Section 101

Closed book examination

Time: 50 minutes

Last Name _____ First _____ Signature _____

Student Number _____

Special Instructions:

No books, notes, or calculators are allowed.

Student Conduct during Examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		15
2		5
3		10
4		10
5		10
Bonus		5
Total		50

1. (15 marks)

(a) (4 marks) Let:

$$f(x, y) = \sin(x^2 + xy) + e^y.$$

Find $\frac{\partial f}{\partial x} \big|_{(0,0)}$ and $\frac{\partial f}{\partial y} \big|_{(0,0)}$. *Simplify your answers.***Solution:** Compute the first order partial derivatives of f :

$$\frac{\partial f}{\partial x}(x, y) = (2x + 1) \cos(x^2 + xy), \quad \frac{\partial f}{\partial y}(x, y) = x \cos(x^2 + xy) + e^y.$$

Evaluate at the point $(0, 0)$:

$$\frac{\partial f}{\partial x} \big|_{(0,0)} = \cos 0 = 1, \quad \frac{\partial f}{\partial y} \big|_{(0,0)} = e^0 = 1.$$

(b) (3 marks) Find all values of a such that $\mathbf{v} = \langle a, -2, -1 \rangle$ is orthogonal to $\mathbf{w} = \langle a, 3, a \rangle$.**Solution:** Since $\mathbf{v} = \langle a, -2, -1 \rangle$ is orthogonal to $\mathbf{w} = \langle a, 3, a \rangle$, we have that:

$$\begin{aligned} 0 &= a(a) + (-2)(3) + (-1)(a) = a^2 - 6 - a = (a - 3)(a + 2) \\ &\Rightarrow a = 3, -2. \end{aligned}$$

(c) (2 marks) Find an equation of the plane \mathcal{P} which has the normal vector $\mathbf{n}_{\mathcal{P}} = \langle -1, 2, 0 \rangle$ and passes through the point $P(-3, 3, 5)$. *Simplify your answer.*

Solution: An equation of the plane \mathcal{P} which has the normal vector $\mathbf{n}_{\mathcal{P}} = \langle -1, 2, 0 \rangle$ and passes through the point $P(-3, 3, 5)$ is:

$$\begin{aligned} -1(x + 3) + 2(y - 3) + 0(z - 5) &= 0 \\ \Rightarrow -x + 2y &= 9. \end{aligned}$$

(d) (3 marks) Determine if the plane \mathcal{P} in part (c) is parallel to the plane \mathcal{Q} given by the equation $4x = 8y$. *Justify your answer.*

Solution: If the plane \mathcal{P} in part (c) is parallel to the plane \mathcal{Q} given by the equation $4x = 8y$, then the normal vector $\mathbf{n}_{\mathcal{P}}$ of \mathcal{P} is parallel to the normal vector $\mathbf{n}_{\mathcal{Q}}$ of \mathcal{Q} . The standard form for \mathcal{Q} is $4x - 8y = 0$, which means $\mathbf{n}_{\mathcal{Q}} = \langle 4, -8, 0 \rangle$. Since

$$\mathbf{n}_{\mathcal{P}} = \langle -1, 2, 0 \rangle = \frac{-1}{4} \langle 4, -8, 0 \rangle = \frac{-1}{4} \mathbf{n}_{\mathcal{Q}},$$

the two normal vectors are parallel, and thus, the two planes are also parallel.

(e) (3 marks) Assume that $f(x, y)$ has continuous partial derivatives of all orders, and

$$f_{yx}(x, y) = e^{xy^2},$$

compute f_{xyy} . State in detail any result that you use.

Solution: Since $f(x, y)$ has continuous partial derivatives of all orders, in particular, f_{xy} and f_{yx} are continuous on \mathbb{R}^2 . By Clairaut's theorem, $f_{xy} = f_{yx}$. Thus,

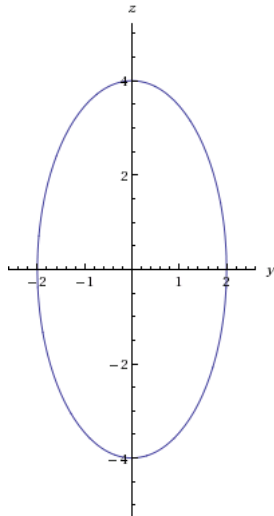
$$f_{xyy} = (f_{xy})_y = (f_{yx})_y = 2xye^{xy^2}.$$

2. (5 marks) Consider the surface S given by:

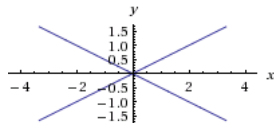
$$z^2 = x^2 - 4y^2.$$

(a) (4 marks) Find and sketch the traces of S in the $x = 4$ and $z = 0$ planes. *Label clearly the axes, and all intercepts on the traces.*

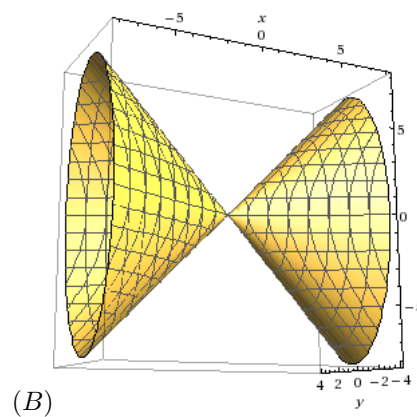
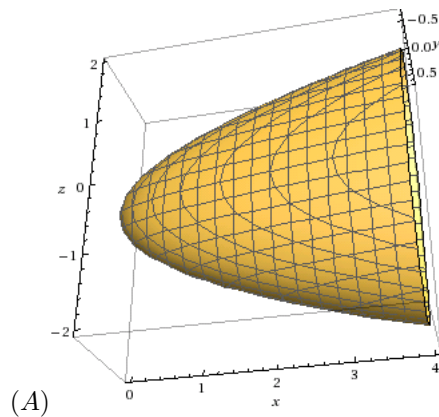
Solution: For $x = 4$, the trace is $z^2 = 16 - 4y^2$, which is equivalent to $z^2 + 4y^2 = 16$ (an ellipse in yz -plane).



For $z = 0$, the trace is $0 = x^2 - 4y^2 = (x - 2y)(x + 2y)$, which consists of two lines in the xy -plane.



(b) (1 mark) Based on the traces you sketched above, which of the following renderings represents the graph of the surface?



Solution: The answer is (B) because in (A) the trace of the surface at $z = 0$ looks like a parabola, not a pair of intersecting lines. (Alternatively, the surface $z^2 = x^2 - 4y^2$ consists points with negative z -values, whereas the surface in (A) does not have any.)

3. (10 marks) Let R be the region $\{x^2 + 4y^2 \leq 4, x \leq 0\}$. Find the maximum and minimum values of the function

$$f(x, y) = 6 - x^2 - 2y^2.$$

on the *boundary of the region R* .

Solution: The boundary of the region R consists of two pieces: the arc which can be parametrized by $x = 2 \cos \theta$ and $y = \sin \theta$ for $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$, and the vertical segment $x = 0$ for $-1 \leq y \leq 1$. We will find the potential candidates where the maximum and minimum can occur on each piece:

- *On the arc:* We have that $f(x, y) = g(\theta) = 6 - (2 \cos \theta)^2 - 2(\sin \theta)^2 = 6 - 4(\cos \theta)^2 - 2(\sin \theta)^2$ for $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. Then, $g'(\theta) = 4 \sin \theta \cos \theta = 0$ if and only if $\theta = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. So, there are 3 points where extrema can occur: $(-2, 0)$ (critical point), $(0, 1)$ and $(0, -1)$ (critical point and also end points).
- *On the vertical segment:* We have that $f(0, y) = h(y) = 6 - 2y^2$ for $-1 \leq y \leq 1$. So, $h'(y) = -4y = 0$ when $y = 0$. So, there are 3 points where extrema can occur: $(0, 0)$ (critical point), $(0, -1)$ and $(0, 1)$ (end points).

Evaluate f at those points, we get:

$$f(-2, 0) = 2, \quad f(0, 1) = f(0, -1) = 4, \quad f(0, 0) = 6$$

Thus, on the boundary of R , f attains the absolute maximum value 6 at the points $(0, 0)$ and the absolute minimum value 2 at the point $(-2, 0)$.

4. (10 marks) Find *all* critical points of the following function:

$$f(x, y) = x^4 + y^4 - 4xy.$$

Classify each point as a local minimum, local maximum, or saddle point.

Solution: Compute the first-order partial derivatives of f :

$$f_x(x, y) = 4x^3 - 4y, \quad f_y(x, y) = 4y^3 - 4x.$$

Since both f_x and f_y are defined at every point in \mathbb{R}^2 , the only critical points of f are those at which $f_x = f_y = 0$. If $f_x = 0$, then $y = x^3$. Replacing $y = x^3$ into $f_y = 0$, we get:

$$4x^9 - 4x = 4x(x^8 - 1) = 0 \Rightarrow x = 0, \pm 1.$$

So, we get three critical points $(0, 0)$, $(-1, -1)$ and $(1, 1)$. Compute the second-order partial derivatives and the discriminants,

$$f_{xx} = 12x^2, \quad f_{yy} = 12y^2, \quad f_{xy} = -4, \quad D(x, y) = 144x^2y^2 - 16$$

Using the Second Derivative Test to classify the points, we get:

- At the point $(0, 0)$, $D(0, 0) = -16 < 0$, so $(0, 0)$ is a saddle point.
- At the point $(1, 1)$, $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, so $(1, 1)$ is a local minimum.
- At the point $(-1, -1)$, $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, so $(-1, -1)$ is also a local minimum.

5. (10 marks) A company produces:

$$P(x, y) = 5x^{\frac{1}{5}}y^{\frac{4}{5}}$$

units of goods per week, utilizing x units of labour and y units of capital. If labour costs \$1 per unit, and capital costs \$1 per 8 units, use the method of Lagrange multiplier to find the most cost-efficient division of labour and capital that the company can adopt if its goal is to produce 80 units of goods per week.

Clearly state the objective function and the constraint. *You are not required to justify that the solution you obtained is the absolute maximum.* **A solution that does not use the method of Lagrange multipliers will receive no credit, even if it is correct.** *Note:* $32 = 2^5$.

Solution: Since labour costs \$1 per unit, and capital costs \$1 per 8 units, the cost function is $C(x, y) = x + y/8$. The objective function to minimize is the cost function $C(x, y) = x + y/8$, and the constraint function is $g(x, y) = 5x^{\frac{1}{5}}y^{\frac{4}{5}} - 80 = 0$ (since its goal is to produce 80 units). Using Lagrange multiplier, we need to solve the following system of equations:

$$\begin{aligned}\nabla C(x, y) &= \lambda \nabla g(x, y) \\ g(x, y) &= 0\end{aligned}$$

More explicitly, we need to solve:

$$\begin{aligned}1 &= \lambda x^{-\frac{4}{5}} y^{\frac{4}{5}} \\ 1/8 &= 4\lambda x^{\frac{1}{5}} y^{-\frac{1}{5}} \\ 5x^{\frac{1}{5}} y^{\frac{4}{5}} - 80 &= 0.\end{aligned}$$

Isolate for λ in the first two equations, we get:

$$\lambda = x^{\frac{4}{5}} y^{-\frac{4}{5}}, \quad \lambda = \frac{1}{32} x^{-\frac{1}{5}} y^{\frac{1}{5}}.$$

Equate the above equations, we get:

$$\begin{aligned}x^{\frac{4}{5}} y^{-\frac{4}{5}} &= \frac{1}{32} x^{-\frac{1}{5}} y^{\frac{1}{5}} \\ 32x &= y.\end{aligned}$$

Replace $32x = y$ in the third equation, we get $80x - 80 = 0$, so $x = 1$. Thus, $y = 32$ and $\lambda = 1/16$. Therefore, if the company uses 1 units of labour and 32 units of capital, then it can produce 80 units of good while minimizing its costs.

Bonus. (5 marks)

Find and sketch the domain of the function $f(x, y) = \arcsin(x^2 + y^2 - 5)$. *Shade the area of the domain, and label all intercepts.*

Solution: In order to compute \arcsin , we need $-1 \leq x^2 + y^2 - 5 \leq 1$, that is, $4 \leq x^2 + y^2 \leq 6$. So, the domain of f is:

$$D = \{(x, y) \in \mathbb{R}^2 \mid 4 \leq x^2 + y^2 \leq 6\}.$$

Graphically, the domain is the area between two circles centered at $(0, 0)$ with radius 2 and $\sqrt{6}$. The circles itself are solid since they are included in the domain.

The End